SYDNEY BOYS HIGH SCHOOL
MOORE PARK, SURRY HILLS

2009
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instruction

- Reading Time – 5 Minutes
- Working time – 180 Minutes
- Write using black or blue pen. Pencil may be used for diagrams.
- Board approved calculators maybe used.
- All necessary working should be shown in every question if full marks are to be awarded.
- Marks may **NOT** be awarded for messy or badly arranged work.
- Start each question in a separate answer booklet.
- Answer in simplest exact form unless otherwise instructed.

Total Marks – 120

- Attempt questions 1-8
- All questions are of equal value

Examiner: *C. Kourtesis*

This is an assessment task only and does not necessarily reflect the content or format of the Higher School Certificate

Question 1. (15 marks) Marks

(a) Find: (i) $\int \frac{1}{\sqrt{x+8}}\,dx$ 3

(ii) $\int \frac{1}{x^2+9}\,dx$

(b) Use integration by parts to find 3

$$\int x \ln x \, dx$$

(c) Use completion of squares to find 2

$$\int \frac{dx}{\sqrt{6-x-x^2}}$$

(d) i) Find real numbers a, b and c such that $\frac{1}{x^2(2-x)} = \frac{ax+b}{x^2} + \frac{c}{2-x}$ 4

ii) Hence evaluate $\int_1^{1.5} \frac{dx}{x^2(2-x)}$

(e) Use the substitution $x = \tan y$ to show that 3

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \frac{\pi+2}{8}$$

Question 2. (15 marks) Marks

(a) If k is a real number and $z = k - 2i$ express $\overline{(iz)}$ in the form $x + iy$ where x and y are real numbers. 2

(b) Solve the equation 2

$$\overline{z} = 3z - 1$$

where $z = x + iy$ $(x,\ y \text{ real})$

(c) On an Argand diagram shade the region specified by both the conditions 3

$$\mathrm{Im}(z) \leq 4 \text{ and } |z - 4 - 5i| \leq 3$$

(d) If $\mathrm{cis}\theta = \cos\theta + \mathrm{i}\sin\theta$ express 2

$$(4\mathrm{cis}\alpha)^2(2\mathrm{cis}\beta)^3$$

in modulus-argument form.

(e) i) Find the equation, in Cartesian form, of the locus of the point z if 4

$$\mathrm{Re}\left(\frac{z}{z+2}\right) = 0$$

ii) Sketch the locus of z satisfying the above.

(f) If α and β are real show that $(\alpha + \beta i)^{2002} + (\beta - \alpha i)^{2002} = 0$. 2

Question 3. (15 marks) Marks

(a) Consider the function 8

$$f(x)=\frac{x^3}{(1-x)^2}$$

i) Show that $f'(x)=\dfrac{x^2[3-x]}{(1-x)^3}$

ii) Use the first derivative $f'(x)$ to determine the nature of the stationary points.

iii) Write down the equations of any asymptotes.

iv) Sketch the graph of $y=f(x)$ showing all essential features.

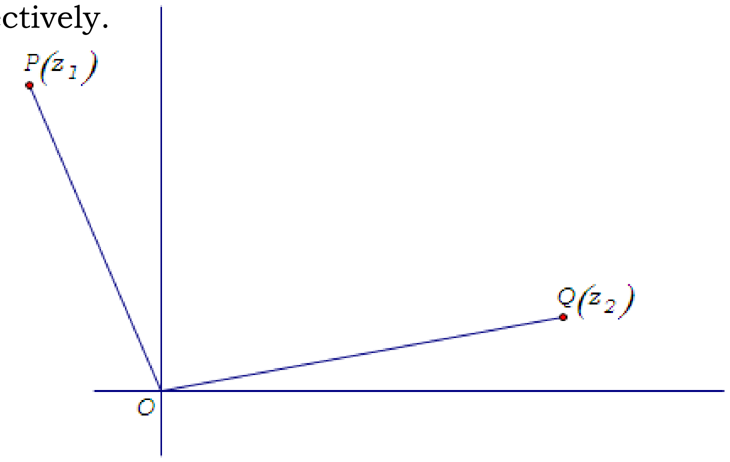
(b) 4

i) Sketch the graphs of $y=\sin x$ and $y=\sqrt{\sin x}$ for $0\le x\le\dfrac{\pi}{2}$ on the same diagram.

ii) Hence show that $1<\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx<\frac{\pi}{2}$

NOTE: You are NOT required to evaluate the integral $\displaystyle\int_0^{\frac{\pi}{2}}\sqrt{\sin x}\,dx$

(c) In the diagram below points P and Q represent the complex numbers z_1 and z_2 respectively. 3



i) Copy the diagram in your examination booklet and indicate the point representing the complex number z_1+z_2

ii) If the length of PQ is $|z_1-z_2|$ and $|z_1-z_2|=|z_1+z_2|$ what can be said about $\dfrac{z_2}{z_1}$

Question 4. (15 marks) Marks

(a) The real cubic polynomial $ax^3+9x^2+ax=30$ has $-3+i$ as a root. 4

i) Show that $x^2+6x+10$ is a quadratic factor of the cubic polynomial.

ii) Show that a = 2.

iii) Write down all the roots of the polynomial.

(b) Show that the polynomial $P(x)=nx^{n+1}-(n+1)x^n+1$ is divisible by $(x-1)^2$ 2

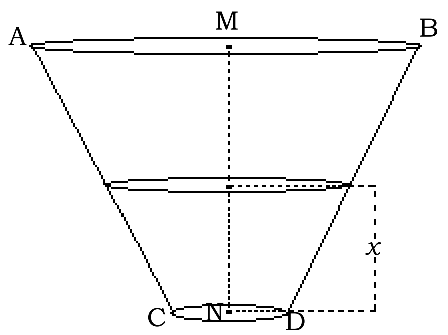
(c) 4

i) Sketch the graphs of $y=\dfrac{1}{x^2+1}$ and $y=\dfrac{1}{x^2+2}$ on the same set of axes.

ii) The area bounded by the two curves in (i) and the ordinates at x = 0 and x = 2 is rotated about the y-axis. Use the cylindrical shell method to show that the volume of the resulting solid is

$$\pi\ln\frac{5}{3}.$$

(d) A drinking glass is in the shape of a truncated cone, in which the internal diameter of the top and bottom are 8cm and 6cm respectively. 5



AB = 8cm
CD = 6cm

i) If the internal height of the glass, MN, is 10cm show that the area of the cross-section x cm above the base is

$$\pi\left(3+\frac{x}{10}\right)^2 \text{ cm}^2.$$

ii) Hence find by integration, the volume of liquid the glass can hold (answer to the nearest mL).

Question 5. (15 marks) Marks

The equation of an ellipse E is given by $\frac{x^2}{9}+\frac{y^2}{5}=1$

i) Find the eccentricity of E 1

ii) Write down the 3
α) coordinates of the foci
β) equations of the directrices
γ) equation of the major auxiliary circle A.

iii) Draw a neat sketch of E showing clearly the features in part ii) 2

iv) A line parallel to the positive y-axis meets the x-axis at N and the curves E, A at P and Q respectively. If N has coordinates $(3\cos\theta, 0)$ find the coordinates of P and Q. [P and Q are in the first quadrant] 2

v) Show that the equations of the tangents at P and Q are $\sqrt{5}x\cos\theta+3y\sin\theta=3\sqrt{5}$ and $x\cos\theta+y\sin\theta=3$ respectively. 4

vi) Show that the point of intersection R of these tangents lies on the major axis of E produced. 1

vii) Prove that $ON \bullet OR$ is independent of the position of P and Q on the curves. 2

Question 6. (15 marks) Marks

(a) i) A particle of mass m falls vertically from rest, from a point o, in a medium whose resistance is mkv, where k is a positive constant and v its velocity after t seconds. 4

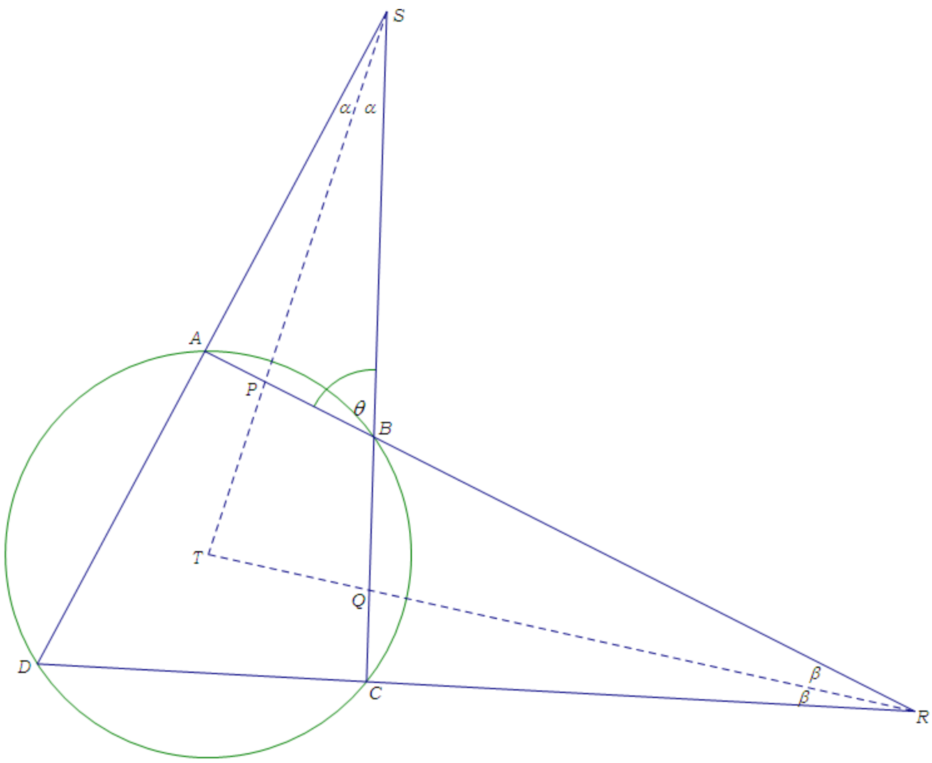
$$\text{Show that } v = \frac{g}{k}\left(1 - e^{-kt}\right)$$

ii) An equal particle is projected vertically upwards with initial velocity U in the same medium. [The particle is released simultaneously with the first particle]. 4

Show that the velocity of the first particle when the second particle is momentarily at rest is given by $\dfrac{VU}{V+U}$

where V is the terminal velocity of the first particle.

(b) 7



$ABCD$ is a cyclic quadrilateral.

The sides AB and CD produced intersect at R and the sides CB and DA produced intersect at S. ST and RT intersect AR and CS at P and Q respectively.

The bisectors of $C\hat{S}D$ and $A\hat{R}D$ meet at T.

Let $A\hat{S}T = B\hat{S}T = \alpha$ and $A\hat{R}T = D\hat{R}T = \beta$ and $A\hat{B}S = \theta$.

i) Show that $T\hat{P}B + T\hat{Q}B = \alpha + \beta + 2\theta$

ii) Prove that ST is perpendicular to RT.

Question 7. (15 marks) Marks

(a) Given that $\tan 5\theta = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$, where $t = \tan\theta$ [Do not prove this] 5

i) Solve the equation $\tan 5\theta = 0$ for $0 \le \theta \le \pi$

ii) Hence prove that

α) $\tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5} = \sqrt{5}$

β) $\tan^2\dfrac{\pi}{5} + \tan^2\dfrac{2\pi}{5} = 10$

(b) i) Show that $\int x\tan^{-1} x\,dx = \dfrac{1}{2}(x^2+1)\tan^{-1} x - \dfrac{1}{2}x + c$ 4

ii) If $u_n = \int_0^1 x^n \tan^{-1} x\,dx$ for $n \ge 2$ show that

$$u_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}u_{n-2}$$

(c) Show that the number of ways in which $2n$ persons may be seated at two round tables, n persons being seated at each is 2

$$\frac{(2n)!}{n^2}$$

(d) i) There are 6 persons from whom a game of tennis is to be made up, two on each side. How many different matches can be arranged if a change in either pair gives a different match? 4

ii) How many different matches are possible if two particular persons are to both play in the match?

Question 8. (15 marks) Marks

(a) Suppose a, b, c and d are positive real numbers. 5

i) Prove that $\dfrac{a}{b}+\dfrac{b}{a}\geq 2$.

ii) Deduce that $\dfrac{a+b+c}{d}+\dfrac{b+c+d}{a}+\dfrac{c+d+a}{b}+\dfrac{d+a+b}{c}\geq 12$.

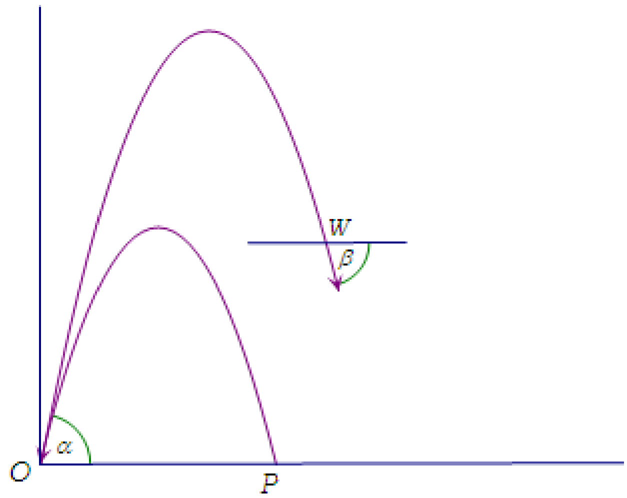
iii) Hence prove that if $a+b+c+d=1$, then:

$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\geq 16.$$

(b) Two stones are thrown simultaneously from the same point O in the same direction and with the same non-zero angle of projection α, but with different velocities U and V ($U < V$). 6

The slower stone hits the ground at a point P on the same level as the point of projection.

At that instant the faster stone is at a point W on its downward path, making an angle β with the horizontal.



i) Show that $V(\tan\alpha+\tan\beta)=2U\tan\alpha$

ii) Deduce that if $\beta=\dfrac{1}{2}\alpha$ then $U<\dfrac{3}{4}V$

(c) i) Show by graphical means that 4

$$\ln ex > e^{-x} \text{ for } x\geq 1$$

ii) Hence, or otherwise, show that

$$\ln(n!e^n) > e^{-n}\left(\frac{e^n-1}{e-1}\right)$$

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1}x^{n+1},\ n \neq -1;\ x \neq 0, \text{if } n < 0$$

$$\int \frac{1}{x} dx = \ln x,\ x > 0$$

$$\int e^{ax} dx = \frac{1}{a}e^{ax},\ a \neq 0$$

$$\int \cos ax\, dx = \frac{1}{a}\sin ax,\ \ a \neq 0$$

$$\int \sin ax\, dx = -\frac{1}{a}\cos ax,\ \ a \neq 0$$

$$\int \sec^2 ax\, dx = \frac{1}{a}\tan ax,$$

$$\int \sec ax \tan ax\, dx = \frac{1}{a}\sec ax,\ \ a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a}\tan^{-1}\frac{x}{a},\ \ a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1}\frac{x}{a},\ \ a > 0,\ \ -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln\left(x + \sqrt{x^2 - a^2}\right),\ x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE: $\ln x = \log_e x,\ \ x > 0$

1) a) i) $\displaystyle\int \frac{1}{\sqrt{x+8}}\,dx = \int (x+8)^{-\frac{1}{2}}\,dx$

$$= \frac{(x+8)^{\frac{1}{2}}}{\frac{1}{2}\cdot 1} + C$$

$$= 2\sqrt{x+8} + C$$

ii) $\displaystyle\int \frac{1}{x^2+9}\,dx = \frac{1}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

$a=3$

b) $\displaystyle\int x\ln x\,dx = \frac{x^2}{2}\ln x - \frac{1}{2}\int x\,dx$

$u = \ln x \quad v' = x$

$u' = \frac{1}{x} \quad v = \frac{x^2}{2}$

$$= \frac{x^2}{2}\ln x - \frac{x^2}{4} + C$$

c) $\displaystyle\int \frac{dx}{\sqrt{6-x-x^2}} = \int \frac{dx}{\sqrt{-(x^2+x+\frac{1}{4})+\frac{25}{4}}}$

$$= \int \frac{dx}{\sqrt{\frac{25}{4}-(x+\frac{1}{2})^2}}$$

$a = \frac{5}{2}$

OR make a substitution

$$= \sin^{-1}\left(\frac{(x+\frac{1}{2})}{(\frac{5}{2})}\right) + C$$

$$= \sin^{-1}\left(\frac{2x+1}{5}\right) + C$$

d) i) $\displaystyle\frac{1}{x^2(2-x)} \equiv \frac{ax+b}{x^2} + \frac{c}{2-x}$

$$1 \equiv (ax+b)(2-x) + cx^2$$

let $x=2$

$1 = c(2)^2$

$c = \frac{1}{4}$

let $x = 0$

$1 = b.2$

$b = \frac{1}{2}$

equate coefficients of x^2

$0 = -a + c$

$a = c$

$\therefore a = \frac{1}{4}$

$a = \frac{1}{4},\ b = \frac{1}{2},\ c = \frac{1}{4}$

ii) $\displaystyle\int_1^{1.5} \frac{dx}{x^2(2-x)} = \int_1^{1.5} \left(\frac{\frac{1}{4}x + \frac{1}{2}}{x^2} + \frac{\frac{1}{4}}{2-x} \right) dx$

$\displaystyle = \int_1^{1.5} \left(\frac{1}{4} \cdot \frac{1}{x} + \frac{1}{2}x^{-2} - \frac{1}{4} \cdot \frac{-1}{2-x} \right) dx$

$\displaystyle = \left[\frac{1}{4}\ln x - \frac{1}{2}x^{-1} - \frac{1}{4}\ln(2-x) \right]_1^{1.5}$

$\displaystyle = \left[-\frac{1}{2x} + \frac{1}{4}\ln\left(\frac{x}{2-x}\right) \right]_1^{1.5}$

$\displaystyle = -\frac{1}{2(1.5)} + \frac{1}{4}\ln\left(\frac{1.5}{2-1.5}\right) - \left(-\frac{1}{2(1)} + \cancel{\frac{1}{4}\ln\left(\frac{1}{2-1}\right)}^{0} \right)$

$\displaystyle = -\frac{1}{3} + \frac{1}{4}\ln 3 + \frac{1}{2}$

$\displaystyle = \frac{1}{4}\ln 3 + \frac{1}{6}$

e) $\displaystyle\int_0^1 \frac{dx}{(x^2+1)^2}$

$x = \tan y$

$\frac{dx}{dy} = \sec^2 y$

$dx = \sec^2 y . dy$

when $x = 1$ $\quad x = 0$

$y = \frac{\pi}{4}$ $\quad y = 0$

$$\int_0^1 \frac{dx}{(x^2+1)^2} = \int_0^{\frac{\pi}{4}} \frac{\sec^2 y \, dy}{(\tan^2 y+1)^2}$$

$$= \int_0^{\frac{\pi}{4}} \frac{\sec^2 y}{(\sec^2 y)^2}\, dy$$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sec^2 y}\, dy$$

$$= \int_0^{\frac{\pi}{4}} \cos^2 y \, dy$$

$$= \int_0^{\frac{\pi}{4}} \left(\frac{1}{2} + \frac{1}{2}\cos 2y\right) dy$$

$$= \left[\frac{1}{2}y + \frac{1}{4}\sin 2y\right]_0^{\frac{\pi}{4}}$$

$$= \frac{1}{2}\left(\frac{\pi}{4}\right) + \frac{1}{4}\sin 2\left(\frac{\pi}{4}\right) - \cancel{\left(\frac{1}{2}(0) + \frac{1}{4}\sin 2(0)\right)}^{\,0}$$

$$= \frac{\pi}{8} + \frac{1}{4}(1)$$

$$= \frac{\pi + 2}{8}$$

Question 2

(a) $z = k - 2i$

$\overline{iz} = \overline{i(k-2i)}$

$= \overline{ik + 2}$

$= 2 - ik$

$x = 2, \ y = -k$

(b) $\bar{z} = 3z - 1$

$x - iy = 3(x + iy) - 1$

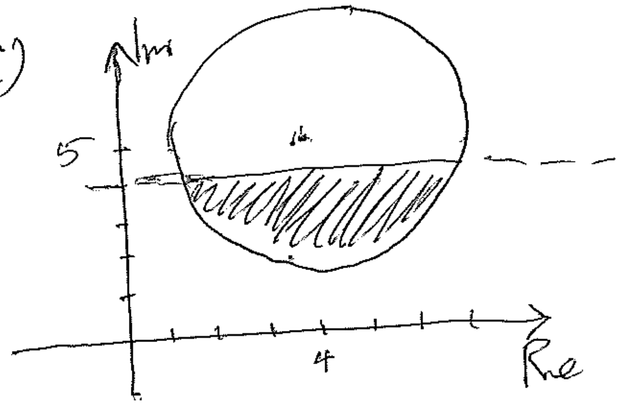
$0 = 2x - 1 + 4iy$

Real: $2x - 1 = 0$ Imag: $4y = 0$

$x = \frac{1}{2}$ $\qquad y = 0$

Solution $z = \frac{1}{2}$

(c)



(d) $(4\,\text{cis}\,\alpha)^2 (2\,\text{cis}\,\beta)^3$

$= 16\,\text{cis}\,2\alpha \cdot 8\,\text{cis}\,3\beta$

$= 128\,\text{cis}(2\alpha + 3\beta)$

(f) Let $z = \alpha + i\beta$

Now $-iz = \beta - i\alpha$

Thus $(\alpha + i\beta)^{2002} + (\beta - i\alpha)^{2002}$

$= z^{2002} + (-iz)^{2002}$

$= z^{2002} + (-1)^{2002} \cdot i^{2002} \cdot z^{2002}$

$= z^{2002} - z^{2002}$

$= 0.$

Q3

(a) $f(x) = \dfrac{x^3}{(1-x)^2}$

(i) $f'(x) = \dfrac{(1-x)^2 . 3x^2 - x^3 . 2(1-x)^1 \times -1}{(1-x)^4}$

$= \dfrac{3x^2(1-x) + 2x^3}{(1-x)^3}$

$= \dfrac{3x^2 - 3x^3 + 2x^3}{(1-x)^3}$

$= \dfrac{3x^2 - x^3}{(1-x)^3}$ ✓

$= \dfrac{x^2(3-x)}{(1-x)^3}$ ✓

(ii) Let $f'(x) = 0$

ie. $x = 0, 3.$

$\therefore y = 0, \frac{27}{4}$

Test $(3, \frac{27}{4})$

Test $(0, 0)$.

x	-1	0	$\frac{1}{2}$
y'	$\frac{1}{2}$	0	5
	/	—	/

$\therefore$ STATIONARY INFLEXION ✓

x	2	3	4
y'	-4	0	$\frac{16}{27}$
	\	—	/

REL. MIN. TURNING PT. ✓

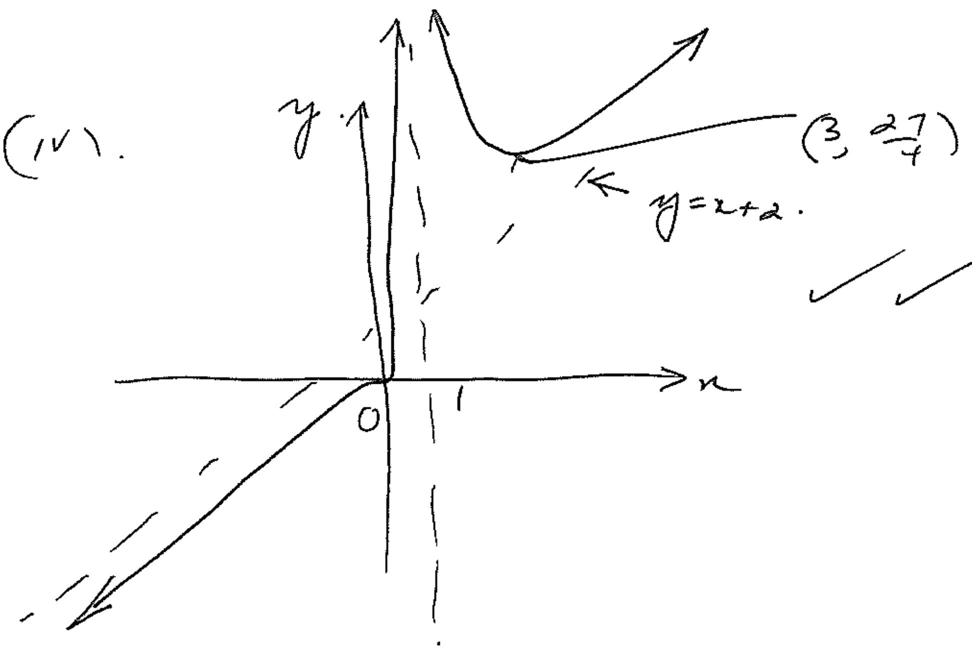
(iii) VERTICAL ASYMPTOTE at $x = 1$ ✓

now $\dfrac{x^3}{x^2-2x+1} = \dfrac{x(x^2-2x+1) + 2(x^2-2x+1) + 3x - 2}{x^2-2x+1}$

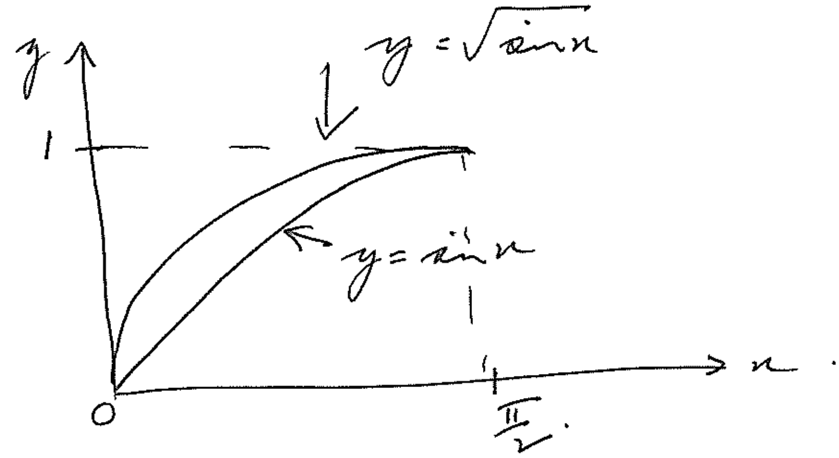
$= x + 2 + \dfrac{3x-2}{x^2-2x+1} \rightarrow x+2$ as $x \rightarrow \infty$.

$\therefore\ y = x+2$ is an oblique asymptote

(iv).



(b)



now $\displaystyle\int_0^{\frac{\pi}{2}} \sin x\,dx < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2}\times 1.$

$$\Big[-\cos x\Big]_0^{\frac{\pi}{2}} < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2}.$$

$$0 - -1 < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2}$$

$$1 < \int_0^{\frac{\pi}{2}} \sqrt{\sin x}\,dx < \frac{\pi}{2}.$$

(CONTD)

(b) For $P(x)$ to be divisible by $(x-1)^2$

then $P(x)=0$ must have a multiple root of degree 2, of value 1.

now $P(1) = n - (n+1) + 1$

$= n - n - 1 + 1$

$= 0$

$P'(x) = n(n+1)x^n - n(n+1)x^{n-1}$

$\therefore P'(1) = n(n+1) - n(n+1)$

$= 0.$

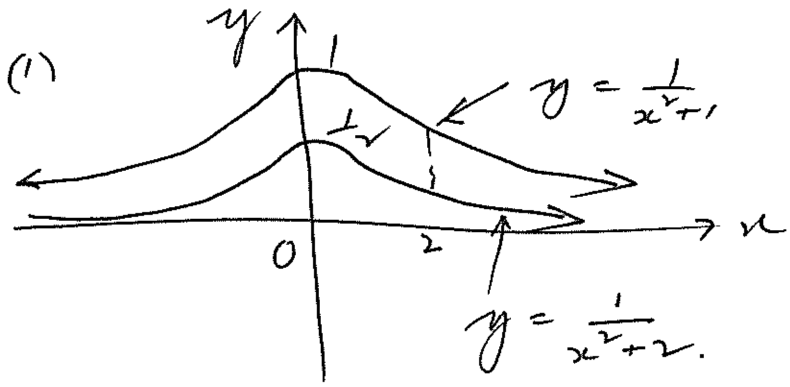
$\therefore P(1) = P'(1) = 0$

$\therefore$ by the multiple root theorem

$x=1$ is a double root.

$\therefore (x-1)^2$ is a factor.

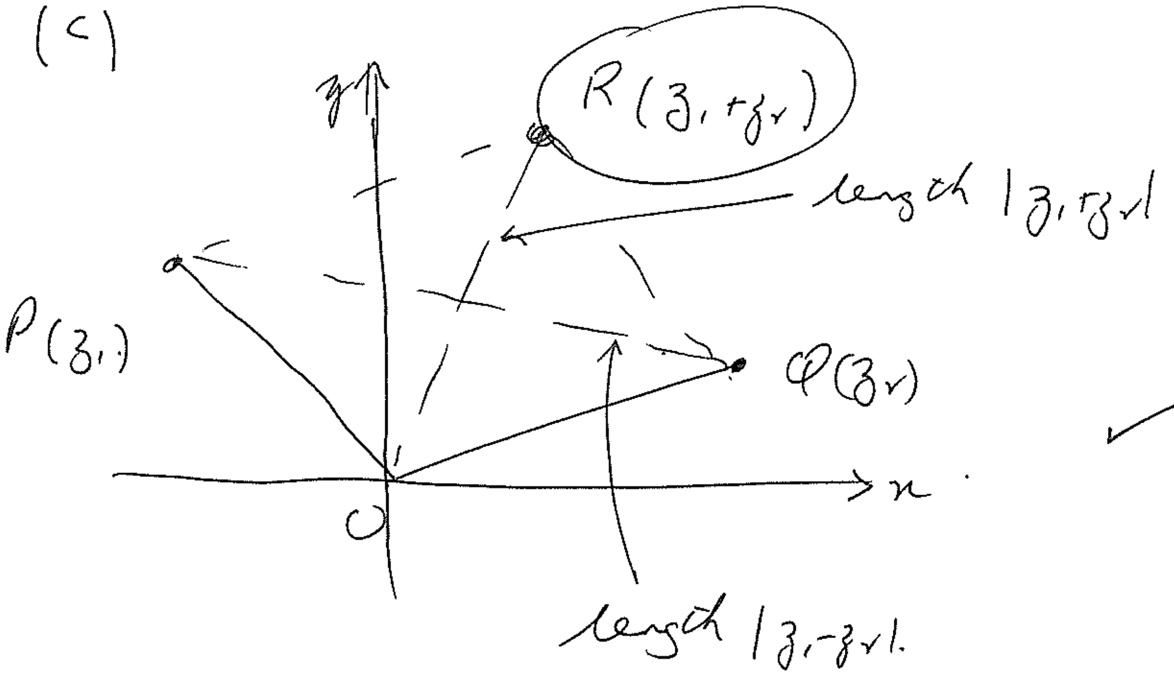
(c) (i)



(ii) $\delta V = 2\pi x \left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right)\delta x$

$V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 2\pi x \left(\frac{1}{x^2+1} - \frac{1}{x^2+2}\right)\delta x$

(c)



now the diagonals of the parallelogram are equal $\therefore$ a ~~rhombus~~. rectangle.

$\therefore \quad z_1 = k i z_2$. OR $z_2 = -k_1 i z_1$

$$\frac{z_2}{z_1} = -k_1 i$$

is IMAGINARY

C(CONT'D)

$$= 2\pi \int_0^2 \left(\frac{x}{x^2+1} - \frac{x}{x^2+2} \right) dx$$

$$= \pi \left[\ln(x^2+1) - \ln(x^2+2) \right]_0^2$$

$$= \pi \left[\ln 5 - \ln 6 - \ln 1 + \ln 2 \right]$$

$$= \pi \ln \frac{10}{6}$$

$$= \boxed{\pi \ln \frac{5}{3} \ u^3}$$

(d). (i) Let the radius of the cross section be r.

$\therefore$ by similarity $\frac{r-3}{x} = \frac{1}{10}$

$r = 3 + \frac{x}{10}$

$\therefore$ Area of the cross-section is $\pi \left(3 + \frac{x}{10}\right)^2$.

$\therefore$ (ii) $V = \lim_{\delta x \to 0} \sum_{x=0}^{10} \pi \left(3 + \frac{x}{10}\right)^2 \delta x$

$$= \pi \int_0^{10} \left(3 + \frac{x}{10}\right)^2 dx$$

$$= \frac{10\pi}{3} \left[\left(3 + \frac{x}{10}\right)^3 \right]_0^{10}$$

$$= \frac{10\pi}{3} \left[4^3 - 3^3 \right]$$

$$= \frac{370\pi}{3} \text{ cc.}$$

$$\fallingdotseq \boxed{387 \text{ ml.}}$$

Q4. (a) (i) Given $ax^3 + 9x^2 + ax - 30 = 0$

with real co-efficients, has a root $-3+i$, it also has $-3-i$ as a root, by the conjugate root theorem.

$\therefore \quad x^2 - (-3+i + -3-i)x + (-3+i)(-3-i)$

is a factor.

i.e. $\boxed{x^2 + 6x + 10}$

(ii) now Clearly $ax^3 + 9x^2 + ax - 30 \equiv$

$(x^2+6x+10)(ax$

now co-eff of x

$LHS = a \quad RHS = 10a - 18$

$\therefore \quad 10a - 18 = a$

$9a = 18$

$\boxed{a = 2}$

(iii) $\sum \alpha_i = -\frac{9}{2} \quad \therefore \quad -3 + i + -3 - i + \alpha = -\frac{9}{2}$

$-6 + \alpha = -\frac{9}{2}$

$\alpha = \frac{3}{2}$

$\therefore$ roots are $\boxed{-3 \pm i, \frac{3}{2}.}$

(b) (see next page)

Solution to Question (5)

[15]

$$\frac{x^2}{9} + \frac{y^2}{5} = 1$$

(i) $b^2 = a^2(1-e^2)$

[1] $5 = 9(1-e^2)$

$e^2 = 4/9 \Rightarrow e = \frac{2}{3}$

(ii) α) $(\pm ae, 0)$

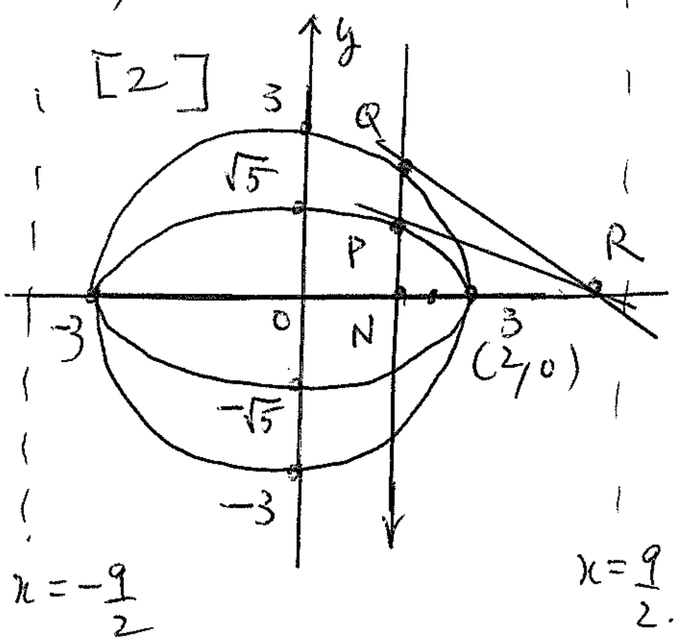
$\therefore (\pm 2, 0)$.

[3] β) $x = \pm \frac{a}{e}$

$= \pm \frac{9}{2}$

γ) $x^2 + y^2 = 9$.

(iii) [2]



(iv) $N : (3\cos\theta, 0)$

[2] $P \; (3\cos\theta, \sqrt{5}\sin\theta)$

$Q \; (3\cos\theta, 3\sin\theta$

(v) At $P \; (3\cos\theta, \sqrt{5}\sin\theta)$

$$\frac{2x}{9} + \frac{2y}{5}\frac{dy}{dx} = 0$$

$$\therefore \frac{dy}{dx} = -\frac{5x}{9y} = -\frac{15\cos\theta}{9\sqrt{5}\sin\theta} = -\frac{\sqrt{5}}{3}\frac{\cos\theta}{\sin\theta}$$

$$\therefore y - \sqrt{5}\sin\theta = -\frac{\sqrt{5}\cos\theta}{3\sin\theta}(x - 3\cos\theta)$$

$(\sin\theta)$ $3y - 3\sqrt{5}\sin^2\theta = -\sqrt{5}x\cos\theta + 3\sqrt{5}\cos^2\theta$

$\therefore \sqrt{5}x\cos\theta + 3y\sin\theta = 3\sqrt{5}$. —— (1)

$$2x + 2y\frac{dy}{dx} = 0$$ [4]

$$\therefore \frac{dy}{dx} = -\frac{\cos\theta}{\sin\theta}$$

$$y - 3\sin\theta = \frac{-\cos\theta}{\sin\theta}(x - 3\cos\theta)$$

$\therefore x\cos\theta + y\sin\theta = 3$. —— (2)

(vi) Solve (1) & (2) $y = 0$

$\therefore x = \frac{3}{\cos\theta} = 3\sec\theta$.

$\therefore R \; (3\sec\theta, 0)$. [1]

(vii) $|ON| = |3\cos\theta|$

$|OR| = |3\sec\theta|$ [2]

$\therefore |ON| \cdot |OR| = \left|3\cancel{\cos\theta} \times \frac{3}{\cancel{\cos\theta}}\right|$

$= 9$

2009 Mathematics Extension 2 Trial HSC: **Questions 7 & 8** solutions

7. (a) Given that $\tan 5\theta = \dfrac{t^5 - 10t^3 + 5t}{5t^4 - 10t^2 + 1}$, where $t = \tan\theta$ [Do not prove this], $\boxed{5}$

i) Solve the equation $\tan 5\theta = 0$ for $0 \leqslant \theta \leqslant \pi$.

Solution:
$$\begin{aligned} \tan 5\theta &= 0, \\ 5\theta &= 0 + n\pi,\ n = 0, 1, 2, 3, \ldots \\ \theta &= \frac{n\pi}{5}, \\ &= 0,\ \frac{\pi}{5},\ \frac{2\pi}{5},\ \frac{3\pi}{5},\ \frac{4\pi}{5},\ \pi. \end{aligned}$$

ii) Hence prove that

α) $\tan\dfrac{\pi}{5}\tan\dfrac{2\pi}{5} = \sqrt{5}$,

Solution: Method 1—
$$\begin{aligned} t^5 - 10t^3 + 5t &= 0, \\ t\big(t^4 - 10t^2 + 5\big) &= 0, \\ t = 0 \text{ or } t^2 &= \frac{10 \pm \sqrt{100 - 20}}{2}, \\ &= 5 \pm 2\sqrt{5}, \\ \text{So } t &= \pm\big(5 \pm 2\sqrt{5}\big). \\ \tan\frac{\pi}{5} &= \sqrt{5 - 2\sqrt{5}}, \qquad \tan\frac{3\pi}{5} = -\sqrt{5 + 2\sqrt{5}}, \\ \tan\frac{2\pi}{5} &= \sqrt{5 + 2\sqrt{5}}, \qquad \tan\frac{4\pi}{5} = -\sqrt{5 - 2\sqrt{5}}, \\ \therefore\ \tan\frac{\pi}{5}\tan\frac{2\pi}{5} &= \sqrt{25 - 20}, \\ &= \sqrt{5}. \end{aligned}$$

Solution: Method 2—
$$\begin{aligned} t^5 - 10t^3 + 5t &= 0, \\ t\big(t^4 - 10t^2 + 5\big) &= 0, \\ t = 0 \text{ or } t^4 - 10t^2 + 5 &= 0. \\ i.e.\ \tan\frac{\pi}{5} \times \tan\frac{2\pi}{5} \times \tan\frac{3\pi}{5} \times \tan\frac{4\pi}{5} &= 5, \text{ (product of roots)} \\ \tan\frac{\pi}{5} \times \tan\frac{2\pi}{5} \times \left(-\tan\frac{2\pi}{5}\right) \times \left(-\tan\frac{\pi}{5}\right) &= 5, \\ i.e.\ \tan^2\frac{\pi}{5} \times \tan^2\frac{2\pi}{5} &= 5, \\ \text{Hence } \tan\frac{\pi}{5}\tan\frac{2\pi}{5} &= \sqrt{5}. \end{aligned}$$
(Positive as both $\frac{\pi}{5}$, and $\frac{2\pi}{5}$ are in the 1st quadrant).

β) $\tan^2\dfrac{\pi}{5} + \tan^2\dfrac{2\pi}{5} = 10$.

Solution: Method 1—
$$\begin{aligned} \tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5} &= 5 - 2\sqrt{5} + 5 + 2\sqrt{5}, \\ &= 10. \end{aligned}$$

Solution: Method 2— (taking roots 2 at a time)

$$\begin{aligned}
-10 &= \tan\frac{\pi}{5}\tan\frac{2\pi}{5} + \tan\frac{\pi}{5}\tan\frac{3\pi}{5} + \tan\frac{\pi}{5}\tan\frac{4\pi}{5} + \tan\frac{2\pi}{5}\tan\frac{3\pi}{5} \\
&\quad + \tan\frac{2\pi}{5}\tan\frac{4\pi}{5} + \tan\frac{3\pi}{5}\tan\frac{4\pi}{5}, \\
&= \tan\frac{\pi}{5}\tan\frac{2\pi}{5} + \tan\frac{\pi}{5}\left(-\tan\frac{2\pi}{5}\right) + \tan\frac{\pi}{5}\left(-\tan\frac{\pi}{5}\right) + \tan\frac{2\pi}{5}\left(-\tan\frac{2\pi}{5}\right) \\
&\quad + \tan\frac{2\pi}{5}\left(-\tan\frac{\pi}{5}\right) + \left(-\tan\frac{2\pi}{5}\right)\left(-\tan\frac{\pi}{5}\right), \\
&= -\tan^2\frac{\pi}{5} - \tan^2\frac{2\pi}{5}, \\
10 &= \tan^2\frac{\pi}{5} + \tan^2\frac{2\pi}{5}.
\end{aligned}$$

(b) i) Show that $\int x\tan^{-1}x\,dx = \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{1}{2}x + c.$ $\boxed{4}$

Solution:

$$\begin{aligned}
\mathrm{I} &= \frac{x^2}{2}\tan^{-1}x - \frac{1}{2}\int\frac{1+x^2-1}{1+x^2}dx, \\
&= \frac{x^2}{2}\tan^{-1}x - \frac{x}{2} + \frac{\tan^{-1}x}{2} + c, \\
&= \frac{1}{2}(x^2+1)\tan^{-1}x - \frac{x}{2} + c.
\end{aligned}
\qquad
\begin{aligned}
u &= \tan^{-1}x, & v' &= x\,dx, \\
u' &= \frac{dx}{1+x^2}, & v &= \frac{x^2}{2}.
\end{aligned}$$

ii) If $u_n = \int_0^1 x^n\tan^{-1}x\,dx$ for $n \geqslant 2$, show that

$$u_n = \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}u_{n-2}.$$

Solution: Method 1—

$$\begin{aligned}
u_n &= \int_0^1 x^n\tan^{-1}x\,dx, \\
&= \left[\frac{x^{n-1}(x^2+1)\tan^{-1}x}{2} - \frac{x^n}{2}\right]_0^1 \\
&\quad - \frac{n-1}{2}\int_0^1 (x^2+1)x^{n-2}\tan^{-1}x\,dx \\
&\quad + \frac{n-1}{2}\int_0^1 x^{n-1}\,dx, \\
&= \frac{\pi}{4} - \frac{1}{2} - \frac{n-1}{2}\int_0^1 x^n\tan^{-1}x\,dx \\
&\quad - \frac{n-1}{2}\int_0^1 x^{n-2}\tan^{-1}x\,dx \\
&\quad + \frac{n-1}{2}\left[\frac{x^n}{n}\right]_0^1, \\
\left(1+\frac{n-1}{2}\right)u_n &= \frac{\pi}{4} - \frac{n}{2n} - \frac{n-1}{2}u_{n-2} + \frac{n-1}{2}.\frac{1}{n}, \\
\frac{n+1}{2}u_n &= \frac{\pi}{4} - \frac{n-1}{2}u_{n-2} - \frac{1}{2n}, \\
u_n &= \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1}u_{n-2}.
\end{aligned}
\qquad
\begin{aligned}
u &= x^{n-1}, \\
u' &= (n-1)x^{n-2}\,dx, \\
v' &= x\tan^{-1}x\,dx, \\
v &= \frac{x^2+1}{2}\tan^{-1}x - \frac{x}{2}.
\end{aligned}$$

Solution: Method 2—

$$\begin{aligned}
u_n &= \int_0^1 x^n \tan^{-1} x\, dx, \\
&= \left[\frac{x^{n-1}\tan^{-1} x}{n+1}\right]_0^1 - \frac{1}{n+1}\int_0^1 \frac{x^{n+1}}{1+x^2}\, dx, \\
&= \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 \frac{(x^2+1)x^{n-1} - x^{n-1}}{1+x^2}\, dx, \\
&= \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 x^{n-1}\, dx + \frac{1}{n+1}\int_0^1 \frac{x^{n-1}}{1+x^2}\, dx, \\
&= \frac{\pi}{4(n+1)} - \frac{1}{n+1}\left[\frac{x^n}{n}\right]_0^1 + \frac{1}{n+1}\left[x^{n-1}\tan^{-1} x\right]_0^1 \\
&\quad - \frac{n-1}{n+1}\int_0^1 x^{n-2}\tan^{-1} x\, dx, \\
&= \frac{\pi}{4(n+1)} - \frac{1}{n(n+1)} + \frac{\pi}{4(n+1)} - \left(\frac{n-1}{n+1}\right) u_{n-2}, \\
&= \frac{\pi}{2(n+1)} - \frac{1}{n(n+1)} - \frac{n-1}{n+1} u_{n-2}.
\end{aligned}$$

$$\begin{aligned}
u &= \tan^{-1}, \\
u' &= \frac{1}{1+x^2}\, dx, \\
v' &= x^n\, dx, \\
v &= \frac{x^{n+1}}{n+1}.
\end{aligned}$$

$$\begin{aligned}
u &= x^{n-1}, \\
u' &= (n-1)x^{n-2}\, dx, \\
v' &= \frac{1}{1+x^2}\, dx, \\
v &= \tan^{-1}.
\end{aligned}$$

(c) Show that the number of ways in which $2n$ persons may be seated at two round tables, n persons being seated at each is $\boxed{2}$

$$\frac{(2n)!}{n^2}.$$

Solution: Ways of choosing people for one table is ${}^{2n}C_n = \dfrac{(2n)!}{(2n-n)!n!}$.

Ways of arranging each table is $(n-1)!$

$$\begin{aligned}
\therefore \text{ Total ways } &= \frac{(2n)!}{n!n!}.(n-1)!(n-1)! \\
&= \frac{(2n)!}{n^2}.
\end{aligned}$$

(d) i) There are 6 persons from whom a game of tennis is to be made up, two on each side. How many different matches can be arranged if a change in either pair gives a different match? $\boxed{4}$

Solution: Ways of choosing 1st pair $= {}^6C_2$,
ways of choosing 2nd pair $= {}^4C_2$.
But pair order not important,

$$\begin{aligned}
\therefore \text{ Number of matches } &= \frac{6!}{4!2!}.\frac{4!}{2!2!}.\frac{1}{2}, \\
&= 45.
\end{aligned}$$

ii) How many different matches are possible if two particular persons are to both play in the match?

Solution: If the two are on the same team,
we only need to choose the other team: ${}^4C_2 = 6$.
If the two are on opposing teams,
(4 ways to get one partner)$\times$(3 ways to get the other) $= 12$.
$\therefore$ Number of matches is $6 + 12 = 18$ altogether.

8. (a) Suppose a, b, c and d are positive real numbers. $\boxed{5}$

i) Prove that $\dfrac{a}{b}+\dfrac{b}{a}\geqslant 2$.

Solution:
$$\begin{aligned}(a-b)^2 &\geqslant 0,\\ a^2-2ab+b^2 &\geqslant 0,\\ a^2+b^2 &\geqslant 2ab,\\ \therefore\ \frac{a}{b}+\frac{b}{a} &\geqslant 2 \text{ as } a,\, b>0.\end{aligned}$$

ii) Deduce that $\dfrac{a+b+c}{d}+\dfrac{b+c+d}{a}+\dfrac{c+d+a}{b}+\dfrac{d+a+b}{c}\geqslant 12$.

Solution: Similarly
$$\begin{aligned}\frac{a}{c}+\frac{c}{a} &\geqslant 2,\\ \frac{a}{d}+\frac{d}{a} &\geqslant 2,\\ \frac{b}{c}+\frac{c}{b} &\geqslant 2,\\ \frac{b}{d}+\frac{d}{b} &\geqslant 2,\\ \frac{c}{d}+\frac{d}{c} &\geqslant 2.\end{aligned}$$

Adding, $\dfrac{b}{a}+\dfrac{c}{a}+\dfrac{d}{a}+\dfrac{a}{b}+\dfrac{c}{b}+\dfrac{d}{b}+\dfrac{a}{c}+\dfrac{b}{c}+\dfrac{d}{c}+\dfrac{a}{d}+\dfrac{b}{d}+\dfrac{c}{d}\geqslant 12,$

i.e. $\dfrac{b+c+d}{a}+\dfrac{a+c+d}{b}+\dfrac{a+b+d}{c}+\dfrac{a+b+c}{d}\geqslant 12.$

iii) Hence prove that if $a+b+c+d=1$, then:

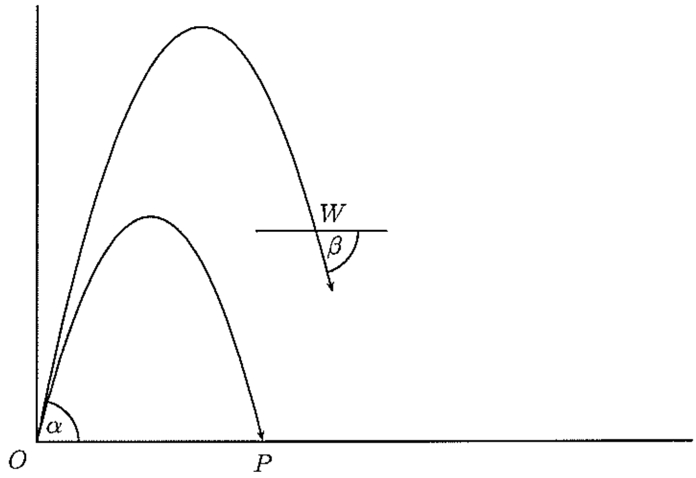
$$\frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d}\geqslant 16.$$

Solution: Now
$$\begin{aligned}a+b+c &= 1-d,\\ a+b+d &= 1-c,\\ a+c+d &= 1-b,\\ b+c+d &= 1-a,\\ \therefore\ \frac{1-a}{a}+\frac{1-b}{b}+\frac{1-c}{c}+\frac{1-d}{d} &\geqslant 12,\\ \frac{1}{a}-1+\frac{1}{b}-1+\frac{1}{c}-1+\frac{1}{d}-1 &\geqslant 12,\\ \text{So } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\frac{1}{d} &\geqslant 16.\end{aligned}$$

(b) Two stones are thrown simultaneously from the same point O in the same direction and with the same non-zero angle of projection α, but with different velocities U and V $(U < V)$. $\boxed{6}$

The slower stone hits the ground at a point P on the same level as the point of projection.

At that instant the faster stone is at a point W on its downward path, making an angle β with the horizontal.



i) Show that $V(\tan\alpha + \tan\beta) = 2U\tan\alpha$.

Solution: For the OP path, For the OW path,

$$\begin{aligned} \ddot{x} &= 0, & \ddot{y} &= -g, & \ddot{x} &= 0, & \ddot{y} &= -g, \\ \dot{x} &= U\cos\alpha, & \dot{y} &= U\sin\alpha - gt, & \dot{x} &= V\cos\alpha, & \dot{y} &= V\sin\alpha - gt, \\ x &= Ut\cos\alpha. & y &= Ut\sin\alpha - g\tfrac{t^2}{2}. & x &= Vt\cos\alpha. & y &= Vt\sin\alpha - g\tfrac{t^2}{2}. \end{aligned}$$

At P, $t = \dfrac{2U\sin\alpha}{g}$.

So at W, $\dot{x} = V\cos\alpha$, $\dot{y} = V\sin\alpha - 2U\sin\alpha$.

$$\begin{aligned} -\tan\beta = \frac{\dot{y}}{\dot{x}} &= \frac{\sin\alpha}{\cos\alpha} - \frac{2U\sin\alpha}{V\cos\alpha}, \\ i.e.\ -V\tan\beta &= V\tan\alpha - 2U\tan\alpha, \\ V(\tan\alpha + \tan\beta) &= 2U\tan\alpha. \end{aligned}$$

ii) Deduce that if $\beta = \dfrac{1}{2}\alpha$, then $U < \dfrac{3}{4}V$.

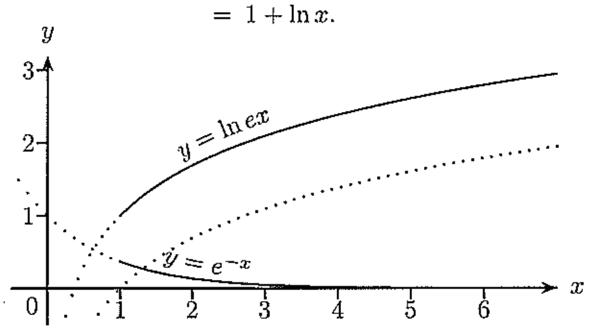
Solution:

$$\begin{aligned} V\left(\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}} + \tan\tfrac{\alpha}{2}\right) &= \frac{2U \times 2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}, \\ V\left(2\tan\tfrac{\alpha}{2} + \tan\tfrac{\alpha}{2} - \tan^3\tfrac{\alpha}{2}\right) &= 4U\tan\tfrac{\alpha}{2}, \\ V\left(3 - \tan^2\tfrac{\alpha}{2}\right) &= 4U, \text{ (as } \tan\tfrac{\alpha}{2} \neq 0) \\ U &= \frac{3V}{4} - \frac{\tan^2\frac{\alpha}{2}}{4}, \\ i.e.\ U &< \frac{3V}{4} \text{ (as } \tan^2\tfrac{\alpha}{2} > 0). \end{aligned}$$

(c) i) Show by graphical means that $\boxed{4}$

$$\ln ex > e^{-x} \text{ for } x \geqslant 1.$$

Solution:
$$\begin{aligned}\ln ex &= \ln e + \ln x,\\ &= 1 + \ln x.\end{aligned}$$



ii) Hence, or otherwise, show that

$$\ln\left(n!e^n\right) > e^{-n}\left(\frac{e^n - 1}{e - 1}\right).$$

Solution:
$$\begin{aligned}\ln n!e^n &= \ln ne.(n-1)e.(n-2)e.(n-3)e\ldots(1)e,\\ &= \ln ne + \ln(n-1)e + \ln(n-2)e + \cdots + \ln e,\\ &> e^{-n} + e^{1-n} + e^{2-n} + \cdots + e^{-1},\\ &> e^{-n}\left(1 + e + e^2 + \cdots + e^{n-1}\right),\\ &> e^{-n}\left(\frac{e^n - 1}{e - 1}\right).\end{aligned}$$